## Exercise 1b)

Assume that the probability of finding a particle at position $x$ in a potential landscape $V(x)$ is given by

$$

P(x) = \frac{e^{-\beta V(x)}}{Z},

$$

where \( V(x) \) is the particle’s potential energy at position $x$ , $\beta = (k\_B T)^{-1}$ where $k\_B$ is the Boltzmann’s constant and \( T \) is the temperature, and \( Z \) is the canonical partition function of the system.

We now want to perform a random walk using these probabilities to determine the movement of the particles. At each step in the random walk, we allow the particle only to move one step to the right, one step to the left, or staying put. We define the probability of moving one step to the right as $ p^+ $, the probability of moving one step to the left as $ p^- $, and the probability of staying put as $ p^0 $. Let the probability of a particle moving to position \*x\* from a position $ x\_0 \in \{x-1, x, x+1\} $ during one time step be proportional to $P(x)$, and assume that the proportionality constant is equal for all step options.

Show that, if the position of the particle before a random walk step is $x\_0$, the probabilities { $p^+, p^0, p^-$} are

$$

p^+ = \frac{1}{1 + e^{-\beta [V(x\_0-1) - V(x\_0+1)]} + e^{-\beta [V(x\_0) - V(x\_0+1)]}},

$$

$$

p^0 = \frac{1}{1 + e^{-\beta [V(x\_0-1) - V(x\_0)]} + e^{-\beta [V(x\_0+1) - V(x\_0)]}},

$$

$$

p^- = \frac{1}{1 + e^{-\beta [V(x\_0+1) - V(x\_0-1)]} + e^{-\beta [V(x\_0) - V(x\_0-1)]}}.

$$

---

$$

Z = e^{-\beta (V(x\_0 - 1))} + e^{-\beta (V(x\_0))} + e^{-\beta (V(x\_0 + 1))}

$$

---

$$

p^+ = \frac{e^{-\beta[V(x\_0 + 1)]}}{Z}

= \frac{e^{-\beta[V(x\_0 + 1)]}}{e^{-\beta [V(x\_0 - 1)]} + e^{-\beta [V(x\_0)]} + e^{-\beta [V(x\_0 + 1)]}}

= \frac{1}{e^{-\beta[V(x\_0 + 1)]}}\frac{1}{e^{-\beta [V(x\_0 - 1)]} + e^{-\beta [V(x\_0)]} + e^{-\beta [V(x\_0 + 1)]}}

$$

$$

= \frac{1}{e^{-\beta [V(x\_0 - 1)-V(x\_0 + 1)]} + e^{-\beta [V(x\_0)-V(x\_0 + 1)]} + e^{-\beta [V(x\_0 + 1)-V(x\_0 + 1)]}}

= \underline{\underline{\frac{1}{1 + e^{-\beta [V(x\_0 - 1)-V(x\_0 + 1)]} + e^{-\beta [V(x\_0)-V(x\_0 + 1)]}}}}

$$

---

$$

p^0 = \frac{e^{-\beta[V(x\_0)]}}{Z}

= \frac{e^{-\beta[V(x\_0)]}}{e^{-\beta [V(x\_0 - 1)]} + e^{-\beta [V(x\_0)]} + e^{-\beta [V(x\_0 + 1)]}}

= \frac{1}{e^{-\beta[V(x\_0)]}}\frac{1}{e^{-\beta [V(x\_0 - 1)]} + e^{-\beta [V(x\_0)]} + e^{-\beta [V(x\_0 + 1)]}}

$$

$$

= \frac{1}{e^{-\beta [V(x\_0 - 1)-V(x\_0)]} + e^{-\beta [V(x\_0)-V(x\_0)]} + e^{-\beta [V(x\_0 + 1)-V(x\_0)]}}

= \underline{\underline{\frac{1}{1 + e^{-\beta [V(x\_0 - 1)-V(x\_0)]} + e^{-\beta [V(x\_0 + 1)-V(x\_0)]}}}}

$$

---

$$

p^- = \frac{e^{-\beta[V(x\_0 - 1)]}}{Z}

= \frac{e^{-\beta[V(x\_0 - 1)]}}{e^{-\beta [V(x\_0 - 1)]} + e^{-\beta [V(x\_0)]} + e^{-\beta [V(x\_0+1)]}}

= \frac{1}{e^{-\beta[V(x\_0 - 1)]}}\frac{1}{e^{-\beta [V(x\_0 - 1)]} + e^{-\beta [V(x\_0)]} + e^{-\beta [V(x\_0 + 1)]}}

$$

$$

= \frac{1}{e^{-\beta [V(x\_0 - 1)-V(x\_0 - 1)]} + e^{-\beta [V(x\_0)-V(x\_0 - 1)]} + e^{-\beta [V(x\_0 + 1)-V(x\_0 - 1)]}}

= \underline{\underline{\frac{1}{1 + e^{-\beta [V(x\_0)-V(x\_0 - 1)]} + e^{-\beta [V(x\_0 + 1)-V(x\_0 - 1)]}}}}

$$

## Exercise 1b)

Assume that the probability of finding a particle at position $x$ in a potential landscape $V(x)$ is given by

$$

P(x) = \frac{e^{-\beta V(x)}}{Z},

$$

where \( V(x) \) is the particle’s potential energy at position $x$ , $\beta = (k\_B T)^{-1}$ where $k\_B$ is the Boltzmann’s constant and \( T \) is the temperature, and \( Z \) is the canonical partition function of the system.

We now want to perform a random walk using these probabilities to determine the movement of the particles. At each step in the random walk, we allow the particle only to move one step to the right, one step to the left, or staying put. We define the probability of moving one step to the right as $ p^+ $, the probability of moving one step to the left as $ p^- $, and the probability of staying put as $ p^0 $. Let the probability of a particle moving to position \*x\* from a position $ x\_0 \in \{x-1, x, x+1\} $ during one time step be proportional to $P(x)$, and assume that the proportionality constant is equal for all step options.

Show that, if the position of the particle before a random walk step is $x\_0$, the probabilities { $p^+, p^0, p^-$} are

$$

p^+ = \frac{1}{1 + e^{-\beta [V(x\_0-1) - V(x\_0+1)]} + e^{-\beta [V(x\_0) - V(x\_0+1)]}},

$$

$$

p^0 = \frac{1}{1 + e^{-\beta [V(x\_0-1) - V(x\_0)]} + e^{-\beta [V(x\_0+1) - V(x\_0)]}},

$$

$$

p^- = \frac{1}{1 + e^{-\beta [V(x\_0+1) - V(x\_0-1)]} + e^{-\beta [V(x\_0) - V(x\_0-1)]}}.

$$

---

\*\*Solution\*\* : We will determine the probabilities $p^+$, $p^-$ and $p^0$ which describes the motion of a particle in a given potential landscape. The probability of finding a particle in a given position follows the Boltzmann distribution:

$$

P(x) = \frac{e^{-\beta V(x)}}{Z},

$$

$Z$ is the canonical partition function of the system that ensures proper normalization (the probabilities sums to 1).

Let’s define the transition probabilities of the particle :

- $p^+$ (move right to $x\_0 + 1$)

- $p^-$ (move left to $x\_0 - 1$)

- $p^0$ (stay at $x\_0$)

$$

p^+ \propto P(x\_0 + 1), \quad p^- \propto P(x\_0 - 1), \quad p^0 \propto P(x\_0)

$$

To normalize these probabilities we define the partition function as:

$$

Z = e^{-\beta (V(x\_0 - 1))} + e^{-\beta (V(x\_0))} + e^{-\beta (V(x\_0 + 1))}

$$

This ensures that the sum of the transition probabilities satisfies:

$$

p^+ + p^0 + p^- = 1

$$

we get:

$$

p^+ = \frac{P(x\_0+1)}{Z}

$$

$$

p^- = \frac{P(x\_0-1)}{Z}

$$

$$

p^0 = \frac{P(x\_0)}{Z}

$$

---

$$

p^+

= \frac{e^{-\beta[V(x\_0 + 1)]}}{Z}

$$

$$

= \frac{e^{-\beta[V(x\_0 + 1)]}}{e^{-\beta [V(x\_0 - 1)]} + e^{-\beta [V(x\_0)]} + e^{-\beta [V(x\_0 + 1)]}}

$$

$$

= \frac{1}{e^{\beta[V(x\_0 + 1)]}}\frac{1}{e^{-\beta [V(x\_0 - 1)]} + e^{-\beta [V(x\_0)]} + e^{-\beta [V(x\_0 + 1)]}}

$$

$$

= \frac{1}{e^{-\beta [V(x\_0 - 1)-V(x\_0 + 1)]} + e^{-\beta [V(x\_0)-V(x\_0 + 1)]} + e^{-\beta [V(x\_0 + 1)-V(x\_0 + 1)]}}

$$

$$

= \underline{\underline{\frac{1}{1 + e^{-\beta [V(x\_0 - 1)-V(x\_0 + 1)]} + e^{-\beta [V(x\_0)-V(x\_0 + 1)]}}}}

$$

---

$$

p^0 = \frac{e^{-\beta[V(x\_0)]}}{Z}

$$

$$

= \frac{e^{-\beta[V(x\_0)]}}{e^{-\beta [V(x\_0 - 1)]} + e^{-\beta [V(x\_0)]} + e^{-\beta [V(x\_0 + 1)]}}

$$

$$

= \frac{1}{e^{\beta[V(x\_0)]}}\frac{1}{e^{-\beta [V(x\_0 - 1)]} + e^{-\beta [V(x\_0)]} + e^{-\beta [V(x\_0 + 1)]}}

$$

$$

= \frac{1}{e^{-\beta [V(x\_0 - 1)-V(x\_0)]} + e^{-\beta [V(x\_0)-V(x\_0)]} + e^{-\beta [V(x\_0 + 1)-V(x\_0)]}}

$$

$$

= \underline{\underline{\frac{1}{1 + e^{-\beta [V(x\_0 - 1)-V(x\_0)]} + e^{-\beta [V(x\_0 + 1)-V(x\_0)]}}}}

$$

---

$$

p^- = \frac{e^{-\beta[V(x\_0 - 1)]}}{Z}

$$

$$

= \frac{e^{-\beta[V(x\_0 - 1)]}}{e^{-\beta [V(x\_0 - 1)]} + e^{-\beta [V(x\_0)]} + e^{-\beta [V(x\_0+1)]}}

$$

$$

= \frac{1}{e^{\beta[V(x\_0 - 1)]}}\frac{1}{e^{-\beta [V(x\_0 - 1)]} + e^{-\beta [V(x\_0)]} + e^{-\beta [V(x\_0 + 1)]}}

$$

$$

= \frac{1}{e^{-\beta [V(x\_0 - 1)-V(x\_0 - 1)]} + e^{-\beta [V(x\_0)-V(x\_0 - 1)]} + e^{-\beta [V(x\_0 + 1)-V(x\_0 - 1)]}}

$$

$$

= \underline{\underline{\frac{1}{1 + e^{-\beta [V(x\_0)-V(x\_0 - 1)]} + e^{-\beta [V(x\_0 + 1)-V(x\_0 - 1)]}}}}

$$

---

Thus, we obtain the final expressions for the transition probabilities:

$$

p^+ = \frac{1}{1 + e^{-\beta [V(x\_0-1) - V(x\_0+1)]} + e^{-\beta [V(x\_0) - V(x\_0+1)]}},

$$

$$

p^0 = \frac{1}{1 + e^{-\beta [V(x\_0-1) - V(x\_0)]} + e^{-\beta [V(x\_0+1) - V(x\_0)]}},

$$

$$

p^- = \frac{1}{1 + e^{-\beta [V(x\_0+1) - V(x\_0-1)]} + e^{-\beta [V(x\_0) - V(x\_0-1)]}}.

$$

These results show that the transition probabilities depend on the differences in potential energy between neighboring positions.

\*\*Exercise 1c)\*\*

What happens to the probabilities {$p^+, p^0, p^−$} given by Eq. (8) when the temperature $T$ satisfies

the following conditions:

- $k\_B T \gg | V(x + 1) - V(x)|$ for any $x$?

- $k\_B T \ll | V(x + 1) - V(x)|$ for any $x$?

$$

$$

An important tool for physicists is identifying the energy scales relevant to what we want to compute. In certain situations, we may ignore some interactions in a problem because the corresponding energy scale can be neglected compared to some other interaction. For example, when you compute

the driving time from Trondheim to Oslo, you can with great precision ignore quantum effects and

relativistic effects such as time dilation, as these effects are small on the relevant energy scale.

How can we simplify the random walk problem using an energy scale argument in the two limits

given above?

---

\*\*Solution\*\*:

Now we examine how transition probabilities behave under different temperature limits and how they influe the nature of the randome walk.

#test

**\*\*Exercise 1c)\*\***

What happens to the probabilities {$p^+, p^0, p^−$} given by Eq. (8) when the temperature $T$ satisfies

the following conditions:

- $k\_B T \gg | V(x + 1) - V(x)|$ for any $x$?

- $k\_B T \ll | V(x + 1) - V(x)|$ for any $x$?

$$

$$

An important tool for physicists is identifying the energy scales relevant to what we want to compute. In certain situations, we may ignore some interactions in a problem because the corresponding energy scale can be neglected compared to some other interaction. For example, when you compute

the driving time from Trondheim to Oslo, you can with great precision ignore quantum effects and

relativistic effects such as time dilation, as these effects are small on the relevant energy scale.

How can we simplify the random walk problem using an energy scale argument in the two limits

given above?

---

**\*\*Solution\*\***:

Now we examine how the transition probabilities behave when the temperature $T$ is very high or low compared to the energy differences in the potential $V(x)$.

$$

$$

**\*\*Case 1: High Temperature ($k\_B T \gg | V(x + 1) - V(x)|$)\*\***

$$

$$

At high temperatures, the thermal energy is much grater than the differences in potential energy. This means that the effect of the potential $V(X)$ is small compared to the probability formulas (Equation (8)). When $T$ is very large, we have:

$$

\beta = \frac{1}{k\_B T} \to 0

$$

Since $\beta$ appears in all the exponential as $e^{-\beta \Delta V}$, this means:

$$

e^{-\beta [V(x\_0+1) - V(x\_0)]} \approx e^{-\beta [V(x\_0-1) - V(x\_0)]} \approx e^{-\beta [V(x\_0)-V(x\_0)]} \approx 1

$$

And the probabilities can be simplefied as:

$$

p^+ = p^0 = p^- = \frac{1}{1+1+1} = \frac{1}{3}

$$

The energy differences $V(x)$ become insignificant because the thermal energy dominates. Therfore the particle moves randomly, and the probabilities are equal.

**\*\*Case 2: Low Temperature ($K\_B T \ll | V(x + 1) - V(x)|$)\*\***

$$

$$

At very low temperatures $ \beta = \frac{1}{k\_B T}$ is very large, making $e^{-\beta \Delta V}$ extremely small for positive energy differences. This means that the probability to transition toward higher energy states become exponentally unlikely.

- if $V(x\_0 + 1) > V(x\_0)$ and , then $e^{-\beta[V(x\_0 + 1) - V(x\_0)]} \to 0$

- if $V(x\_0 + 1) < V(x\_0)$, then $e^{-\beta[V(x\_0) - V(x\_0 + 1)]}\gg$ 1

Nyeste:

**## Exercise 1b)**

Assume that the probability of finding a particle at position $x$ in a potential landscape $V(x)$ is given by

$$

P(x) = \frac{e^{-\beta V(x)}}{Z},

$$

where \( V(x) \) is the particle’s potential energy at position $x$ , $\beta = (k\_B T)^{-1}$ where $k\_B$ is the Boltzmann’s constant and \( T \) is the temperature, and \( Z \) is the canonical partition function of the system.

We now want to perform a random walk using these probabilities to determine the movement of the particles. At each step in the random walk, we allow the particle only to move one step to the right, one step to the left, or staying put. We define the probability of moving one step to the right as $ p^+ $, the probability of moving one step to the left as $ p^- $, and the probability of staying put as $ p^0 $. Let the probability of a particle moving to position *\*x\** from a position $ x\_0 \in \{x-1, x, x+1\} $ during one time step be proportional to $P(x)$, and assume that the proportionality constant is equal for all step options.

Show that, if the position of the particle before a random walk step is $x\_0$, the probabilities { $p^+, p^0, p^-$} are

$$

p^+ = \frac{1}{1 + e^{-\beta [V(x\_0-1) - V(x\_0+1)]} + e^{-\beta [V(x\_0) - V(x\_0+1)]}},

$$

$$

p^0 = \frac{1}{1 + e^{-\beta [V(x\_0-1) - V(x\_0)]} + e^{-\beta [V(x\_0+1) - V(x\_0)]}}, \tag{8}

$$

$$

p^- = \frac{1}{1 + e^{-\beta [V(x\_0+1) - V(x\_0-1)]} + e^{-\beta [V(x\_0) - V(x\_0-1)]}}.

$$

---

**\*\*Solution\*\*** : We will determine the probabilities $p^+$, $p^-$ and $p^0$ which describes the motion of a particle in a given potential landscape. The probability of finding a particle in a given position follows the Boltzmann distribution:

$$

P(x) = \frac{e^{-\beta V(x)}}{Z},

$$

$Z$ is the canonical partition function of the system that ensures proper normalization (the probabilities sums to 1).

Let’s define the transition probabilities of the particle :

- $p^+$ (move right to $x\_0 + 1$)

- $p^-$ (move left to $x\_0 - 1$)

- $p^0$ (stay at $x\_0$)

$$

p^+ \propto P(x\_0 + 1), \quad p^- \propto P(x\_0 - 1), \quad p^0 \propto P(x\_0)

$$

To normalize these probabilities we define the partition function as:

$$

Z = e^{-\beta (V(x\_0 - 1))} + e^{-\beta (V(x\_0))} + e^{-\beta (V(x\_0 + 1))}

$$

This ensures that the sum of the transition probabilities satisfies:

$$

p^+ + p^0 + p^- = 1

$$

we get:

$$

p^+ = \frac{P(x\_0+1)}{Z}

$$

$$

p^- = \frac{P(x\_0-1)}{Z}

$$

$$

p^0 = \frac{P(x\_0)}{Z}

$$

---

$$

p^+

= \frac{e^{-\beta[V(x\_0 + 1)]}}{Z}

$$

$$

= \frac{e^{-\beta[V(x\_0 + 1)]}}{e^{-\beta [V(x\_0 - 1)]} + e^{-\beta [V(x\_0)]} + e^{-\beta [V(x\_0 + 1)]}}

$$

$$

= \frac{1}{e^{\beta[V(x\_0 + 1)]}}\frac{1}{e^{-\beta [V(x\_0 - 1)]} + e^{-\beta [V(x\_0)]} + e^{-\beta [V(x\_0 + 1)]}}

$$

$$

= \frac{1}{e^{-\beta [V(x\_0 - 1)-V(x\_0 + 1)]} + e^{-\beta [V(x\_0)-V(x\_0 + 1)]} + e^{-\beta [V(x\_0 + 1)-V(x\_0 + 1)]}}

$$

$$

= \underline{\underline{\frac{1}{1 + e^{-\beta [V(x\_0 - 1)-V(x\_0 + 1)]} + e^{-\beta [V(x\_0)-V(x\_0 + 1)]}}}}

$$

---

$$

p^0 = \frac{e^{-\beta[V(x\_0)]}}{Z}

$$

$$

= \frac{e^{-\beta[V(x\_0)]}}{e^{-\beta [V(x\_0 - 1)]} + e^{-\beta [V(x\_0)]} + e^{-\beta [V(x\_0 + 1)]}}

$$

$$

= \frac{1}{e^{\beta[V(x\_0)]}}\frac{1}{e^{-\beta [V(x\_0 - 1)]} + e^{-\beta [V(x\_0)]} + e^{-\beta [V(x\_0 + 1)]}}

$$

$$

= \frac{1}{e^{-\beta [V(x\_0 - 1)-V(x\_0)]} + e^{-\beta [V(x\_0)-V(x\_0)]} + e^{-\beta [V(x\_0 + 1)-V(x\_0)]}}

$$

$$

= \underline{\underline{\frac{1}{1 + e^{-\beta [V(x\_0 - 1)-V(x\_0)]} + e^{-\beta [V(x\_0 + 1)-V(x\_0)]}}}}

$$

---

$$

p^- = \frac{e^{-\beta[V(x\_0 - 1)]}}{Z}

$$

$$

= \frac{e^{-\beta[V(x\_0 - 1)]}}{e^{-\beta [V(x\_0 - 1)]} + e^{-\beta [V(x\_0)]} + e^{-\beta [V(x\_0+1)]}}

$$

$$

= \frac{1}{e^{\beta[V(x\_0 - 1)]}}\frac{1}{e^{-\beta [V(x\_0 - 1)]} + e^{-\beta [V(x\_0)]} + e^{-\beta [V(x\_0 + 1)]}}

$$

$$

= \frac{1}{e^{-\beta [V(x\_0 - 1)-V(x\_0 - 1)]} + e^{-\beta [V(x\_0)-V(x\_0 - 1)]} + e^{-\beta [V(x\_0 + 1)-V(x\_0 - 1)]}}

$$

$$

= \underline{\underline{\frac{1}{1 + e^{-\beta [V(x\_0)-V(x\_0 - 1)]} + e^{-\beta [V(x\_0 + 1)-V(x\_0 - 1)]}}}}

$$

---

Thus, we obtain the final expressions for the transition probabilities:

$$

p^+ = \frac{1}{1 + e^{-\beta [V(x\_0-1) - V(x\_0+1)]} + e^{-\beta [V(x\_0) - V(x\_0+1)]}},

$$

$$

p^0 = \frac{1}{1 + e^{-\beta [V(x\_0-1) - V(x\_0)]} + e^{-\beta [V(x\_0+1) - V(x\_0)]}},

$$

$$

p^- = \frac{1}{1 + e^{-\beta [V(x\_0+1) - V(x\_0-1)]} + e^{-\beta [V(x\_0) - V(x\_0-1)]}}.

$$

These results show that the transition probabilities depend on the differences in potential energy between neighboring positions.

c)

**## Exercise 1c)**

What happens to the probabilities {$p^+, p^0, p^−$} given by Eq. (8) when the temperature $T$ satisfies

the following conditions:

- $k\_B T \gg | V(x + 1) - V(x)|$ for any $x$?

- $k\_B T \ll | V(x + 1) - V(x)|$ for any $x$?

$$

$$

An important tool for physicists is identifying the energy scales relevant to what we want to compute. In certain situations, we may ignore some interactions in a problem because the corresponding energy scale can be neglected compared to some other interaction. For example, when you compute

the driving time from Trondheim to Oslo, you can with great precision ignore quantum effects and

relativistic effects such as time dilation, as these effects are small on the relevant energy scale.

How can we simplify the random walk problem using an energy scale argument in the two limits

given above?

---

**\*\*Solution\*\***:

Now we examine how the transition probabilities behave when the temperature $T$ is very high or low compared to the energy differences in the potential $V(x)$.

$$

$$

**### High Temperature ($k\_B T \gg | V(x + 1) - V(x)|$)\*\***

$$

$$

At high temperatures, the thermal energy is much greater than the differences in potential energy. This means that the effect of the potential $V(X)$ is small compared to the probability formulas (Equation (8)). When $T$ is very large, we have:

$$

\beta = \frac{1}{k\_B T} \to 0

$$

Since $\beta$ appears in all the exponential as $e^{-\beta \Delta V}$, this means:

$$

e^{-\beta [V(x\_0+1) - V(x\_0)]} \approx e^{-\beta [V(x\_0-1) - V(x\_0)]} \approx e^{-\beta [V(x\_0)-V(x\_0)]} \approx 1

$$

And the probabilities can be simplefied as:

$$

p^+ = p^0 = p^- = \frac{1}{1+1+1} = \frac{1}{3}

$$

The energy differences $V(x)$ become insignificant because the thermal energy dominates. Therfore the particle moves randomly, and the probabilities are equal.

**### Low Temperature ($K\_B T \ll | V(x + 1) - V(x)|$)**

$$

$$

At very low temperatures $ \beta = \frac{1}{k\_B T}$ is very large, making $e^{-\beta \Delta V}$ extremely small for positive energy differences. This means that the probability to transition toward higher energy states become exponentally unlikely. To analyze the behavior of the probabilities $p^+, p^0, p^-$ in the low-temperature limit, we examine the system by considering representative cases for potential differences.

**\*\*Case 1:\*\*** $V(x + 1) > V(x)$ and $V(x - 1) > V(x + 1)$:

$$

p^+ = \frac{1}{1 + e^{-\beta [V(x\_0-1) - V(x\_0+1)]} + e^{-\beta [V(x\_0) - V(x\_0+1)]}} \approx \frac{1}{1 + e^{-\infty} + e^{\infty}} = \frac{1}{1 + 0 + e^{\infty}} = 0

$$

$$

p^0 = \frac{1}{1 + e^{-\beta [V(x\_0-1) - V(x\_0)]} + e^{-\beta [V(x\_0+1) - V(x\_0)]}} \approx \frac{1}{1 + e^{-\infty} + e^{-\infty}} = \frac{1}{1 + 0 + 0} = 1

$$

$$

p^- = \frac{1}{1 + e^{-\beta [V(x\_0+1) - V(x\_0-1)]} + e^{-\beta [V(x\_0) - V(x\_0-1)]}} \approx \frac{1}{1 + e^{\infty} + e^{\infty}} = 0

$$

**##### Detailed Calculation of case 1 ($p^+$):**

$$

V(x + 1) > V(x) \to V(x + 1) - V(x) > 0 \to V(x) - V(x + 1) < 0

$$

$$

V(x - 1) > V(x + 1) \to V(x - 1) - V(x + 1) > 0 \to V(x + 1) - V(x - 1) < 0

$$

$$

V(x - 1) > V(x) \to V(x - 1) - V(x) > 0 \to V(x) - V(x - 1) < 0

$$

$$

p^+ = \frac{1}{1 + e^{-\beta [V(x\_0-1) - V(x\_0+1)]} + e^{-\beta [V(x\_0) - V(x\_0+1)]}}

$$

Step by step:

$$

e^{-\beta [V(x\_0-1) - V(x\_0+1)]} = e^{-\infty}

$$

Since $V(x - 1) - V(x + 1) > 0$, we get $e^{-\beta [V(x\_0-1) - V(x\_0+1)]} \ll 0$ and $e^{-\infty} \to 0$

---

$$

e^{-\beta [V(x\_0) - V(x\_0+1)]} = e^{\infty}

$$

Since $V(x) - V(x + 1) < 0$, $e^{-\beta [V(x\_0) - V(x\_0+1)]} \gg 0$ meaning the exponent is a lagre positive number, the exponential term grows to $\infty$.

---

Therefore:

$$

p^+ \approx \frac{1}{1 + e^{-\infty} + e^{\infty}} = \frac{1}{1 + 0 + e^{\infty}} = 0

$$

The denominator contains a term that goes to $\infty$, the fraction itself goes to $0$, leading $p^+ \approx 0$

For this specific case, both $p^+$ and $p^-$ become exponentally small, meaning the particle remains at $x\_0$ with probabiility $p^0 \approx 1$ in the low-temerature limit. If the potential conditions were different, the probability of moving in a particular direction would increase accordingly, as transitions are governed by the relative energy differences between neighboring states. For example if we change the conditions to $V(x) > V(x + 1)$ and $V(x - 1) > V(x + 1)$, we get:

$$

p^+ \approx 1

p^0 \approx 0

p^- \approx 0

$$

So regardless of how the potential conditions are changed, at low temperatures ($K\_B T \ll | V(x + 1) - V(x)|$) the probabilties are always determined by the relative energy differences between neighboring states. In each scenario, the transition with the lowest energy cost dominates, meaning its probability will approach 1, while the probabilities of the other two transitions will be exponentially suppressed and approach 0.

$$

$$

This ensures that, regardless of how the potential conditions changes, the system will always overwhelmingly favor a single transition, while the other two remains exponentially suppressed.

Ny

**## Exercise 1c)**

What happens to the probabilities {$p^+, p^0, p^−$} given by Eq. (8) when the temperature $T$ satisfies

the following conditions:

- $k\_B T \gg | V(x + 1) - V(x)|$ for any $x$?

- $k\_B T \ll | V(x + 1) - V(x)|$ for any $x$?

$$

$$

An important tool for physicists is identifying the energy scales relevant to what we want to compute. In certain situations, we may ignore some interactions in a problem because the corresponding energy scale can be neglected compared to some other interaction. For example, when you compute

the driving time from Trondheim to Oslo, you can with great precision ignore quantum effects and

relativistic effects such as time dilation, as these effects are small on the relevant energy scale.

How can we simplify the random walk problem using an energy scale argument in the two limits

given above?

---

**\*\*Solution:\*\***

Now we examine how the transition probabilities behave when the temperature $T$ is very high or low compared to the energy differences in the potential $V(x)$.

$$

$$

**#### High Temperature ($k\_B T \gg | V(x + 1) - V(x)|$)**

$$

$$

At high temperatures, the thermal energy is much greater than the differences in potential energy. This means that the effect of the potential $V(X)$ is small compared to the probability formulas (Equation (8)). When $T$ is very large, we have:

$$

\beta = \frac{1}{k\_B T} \to 0

$$

Since $\beta$ appears in all the exponential as $e^{-\beta \Delta V}$, this means:

$$

e^{-\beta [V(x\_0+1) - V(x\_0)]} \approx e^{-\beta [V(x\_0-1) - V(x\_0)]} \approx e^{-\beta [V(x\_0)-V(x\_0)]} \approx 1

$$

And the probabilities can be simplefied as:

$$

p^+ = p^0 = p^- = \frac{1}{1+1+1} = \frac{1}{3}

$$

The energy differences $V(x)$ become insignificant because the thermal energy dominates. Therfore the particle moves randomly, and the probabilities are equal.

**#### Low Temperature ($K\_B T \ll | V(x + 1) - V(x)|$)**

$$

$$

At very low temperatures $ \beta = \frac{1}{k\_B T}$ is very large, making $e^{-\beta \Delta V}$ extremely small for positive energy differences. This means that the probability to transition toward higher energy states become exponentally unlikely. To analyze the behavior of the probabilities $p^+, p^0, p^-$ in the low-temperature limit, we examine the system by considering representative cases for potential differences.

**\*\*Case 1:\*\*** $V(x + 1) > V(x)$ and $V(x - 1) > V(x + 1)$:

$$

p^+ = \frac{1}{1 + e^{-\beta [V(x\_0-1) - V(x\_0+1)]} + e^{-\beta [V(x\_0) - V(x\_0+1)]}} \approx \frac{1}{1 + e^{-\infty} + e^{\infty}} = \frac{1}{1 + 0 + e^{\infty}} = 0

$$

$$

p^0 = \frac{1}{1 + e^{-\beta [V(x\_0-1) - V(x\_0)]} + e^{-\beta [V(x\_0+1) - V(x\_0)]}} \approx \frac{1}{1 + e^{-\infty} + e^{-\infty}} = \frac{1}{1 + 0 + 0} = 1

$$

$$

p^- = \frac{1}{1 + e^{-\beta [V(x\_0+1) - V(x\_0-1)]} + e^{-\beta [V(x\_0) - V(x\_0-1)]}} \approx \frac{1}{1 + e^{\infty} + e^{\infty}} = 0

$$

**##### Detailed Calculation of case 1 ($p^+$):**

$$

V(x + 1) > V(x) \to V(x + 1) - V(x) > 0 \to V(x) - V(x + 1) < 0

$$

$$

V(x - 1) > V(x + 1) \to V(x - 1) - V(x + 1) > 0 \to V(x + 1) - V(x - 1) < 0

$$

$$

V(x - 1) > V(x) \to V(x - 1) - V(x) > 0 \to V(x) - V(x - 1) < 0

$$

$$

p^+ = \frac{1}{1 + e^{-\beta [V(x\_0-1) - V(x\_0+1)]} + e^{-\beta [V(x\_0) - V(x\_0+1)]}}

$$

Step by step:

$$

e^{-\beta [V(x\_0-1) - V(x\_0+1)]} = e^{-\infty}

$$

Since $V(x - 1) - V(x + 1) > 0$, we get $e^{-\beta [V(x\_0-1) - V(x\_0+1)]} \ll 0$ and $e^{-\infty} \to 0$

---

$$

e^{-\beta [V(x\_0) - V(x\_0+1)]} = e^{\infty}

$$

Since $V(x) - V(x + 1) < 0$, $e^{-\beta [V(x\_0) - V(x\_0+1)]} \gg 0$ meaning the exponent is a lagre positive number, the exponential term grows to $\infty$.

---

Therefore:

$$

p^+ \approx \frac{1}{1 + e^{-\infty} + e^{\infty}} = \frac{1}{1 + 0 + e^{\infty}} = 0

$$

The denominator contains a term that goes to $\infty$, the fraction itself goes to $0$, leading $p^+ \approx 0$

For this specific case, both $p^+$ and $p^-$ become exponentally small, meaning the particle remains at $x\_0$ with probabiility $p^0 \approx 1$ in the low-temerature limit. If the potential conditions were different, the probability of moving in a particular direction would increase accordingly, as transitions are governed by the relative energy differences between neighboring states. For example if we change the conditions to $V(x) > V(x + 1)$ and $V(x - 1) > V(x + 1)$, we get:

$$

p^+ \approx 1

p^0 \approx 0

p^- \approx 0

$$

---

Since transitions to higher energy states are exponentally suppressed in the low-temperature limit, the particle will always move towards the neighboring site with the lowest potential energy. This ensures that, at sufficiently low temperatures, the particle follows an energy-minimizing trajectory, and once it reaches a local minimum, it remains trapped there.

So regardless of how the potential conditions are changed, at low temperatures ($K\_B T \ll | V(x + 1) - V(x)|$) the probabilties are always determined by the relative energy differences between neighboring states. In each scenario, the transition with the lowest energy cost dominates, meaning its probability will approach 1, while the probabilities of the other two transitions will be exponentially suppressed and approach 0.

$$

$$

This ensures that, regardless of how the potential conditions changes, the system will always overwhelmingly favor a single transition, while the other two remains exponentially suppressed.